

# Supergravity as a Yang-Mills theory<sup>1</sup>

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**Abstract:** We give a simple introduction to ordinary and conformal supergravity, and write their actions as squares of curvatures.

## 1 Introduction

Supergravity is the nonabelian gauge theory of supersymmetry. It was constructed in 1976 [1, 2], soon after rigid supersymmetry had been constructed in the early 1970's [3]. There already exist many books and reviews on this subject, so in this contribution we shall not try to give a systematic account of supergravity, but rather put the work of Yang and Mills central, and focus on the similarities and differences between supergravity and Yang-Mills theory. Let us only mention a few of the successes of supergravity.

- It is a complete classical theory in the same way as general relativity, with action and well-understood geometry, and forms the low-energy limit of string theory where such complete results have not yet been found
- it allows a proof of the positive energy conjecture in general relativity
- it has given relations between 5-dimensional tree supergravity and 4-dimensional non-perturbative quantum superYang-Mills theory (the AdS/CFT correspondence)
- it has made the phenomenology of the search for supersymmetric particles at LHC possible, because it can remove the huge cosmological constant of spontaneously broken rigid supersymmetry and the gravitino can eat a Goldstino, explaining why no Goldstino has been seen

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<sup>1</sup>Contribution to “50 Years of Yang-Mills Theory”, World Pub. Co., G. ‘t Hooft editor.

- it has led to various dualities in field theory which one later attempted to extend to string theory
- the unique 11-dimensional supergravity theory provides the only concrete information about the hypothetical M-theory which is supposed to describe all non-perturbative string theory including solitons
- it rephrases differential geometry in terms of Killing spinors instead of Killing vectors. This approach is much more powerful and has led to breakthroughs in various areas of mathematics.

Yang-Mills theory, the gauge theory of internal nonabelian symmetries, has become in the 20-th century what Maxwell theory was in the 19-th century. Its renormalization by 't Hooft and Veltman has led to a consistent quantum gauge field theory, whose radiative corrections are computed and measured in the large laboratories all over the world. The results confirm the theory to incredible precision. As the underlying theory for QCD and electroweak forces, it has resolved the problems of earlier approaches such as the  $V - A$  theory of the weak interactions, or the one-boson exchange models, bootstrap models, dispersion relation approaches, Regge models, etc. for the strong interactions. A theoretical foundation for the Standard Model has been established through the work of Yang, Mills, 't Hooft, Veltman, and Faddeev, Popov, Fradkin, Tyurin, Feynman, Gell-Mann, Bryce DeWitt, Mandelstam, Slavnov, Taylor, Zinn-Justin, B. Lee, Gross, Wilczek, Politzer, Becchi, Rouet, Stora, Tyutin, Nambu, Goldstone, Higgs, Brout, Englert, Bouchiat, Iliopoulos, Meyer, and many, many others. The establishment of nonabelian gauge theory at the classical and quantum level ranks with the discoveries of special and general relativity and quantum mechanics as one of the great achievements of physics.

I have had the privilege of spending the beginning of my scientific life in Utrecht with Tini Veltman, Gerard 't Hooft, Bernard de Wit, Hans Reif, and then, after some postdocs, the rest of my scientific life at Stony Brook with Frank Yang. The many discussions with them in past and present times have revealed to me the personal side of their great discoveries, the uncertainties and worries, but also the satisfaction of just doing interesting work, and the slow realization that something important was being constructed. The

friendship with them has been and still is a continuing source of support for my own activities. Bob Mills I only met at the retirement symposium of Frank in 1999. He struck me as a very decent and honest person. Unfortunately he died soon after.

Yang and Mills wrote their pivotal paper in 1954 without any reference to gravity [4]. However, already in the 1920's the ideas of gauge theory were developed in the context of gravity, notably by Weyl, and we shall connect these two approaches. We shall show that one can also apply the gauge field formalism of Yang and Mills to gravity and supergravity, with an action quadratic in curvatures instead of the linear Einstein-Hilbert action. Of course Weyl in addition considered a locally scale invariant formulation of gravity in one of his earlier papers, in order to explain the meaning of electromagnetism, and this approach (with an extra factor of  $i$  later added when quantum mechanics was discovered) led to the modern concept of gauge symmetry. Also in supergravity such a theory exists. It is called conformal supergravity, to distinguish it from the ordinary theories of supergravity which one might call Poincaré (or rather super Poincaré) theories. Conformal supergravity has also an  $R^2$  action, very much of the type of Yang and Mills, but one needs constraints on curvatures and torsions. Ordinary (and also conformal) supergravity can be very beautifully written in superspace [5, 6] but then one also needs constraints on the supercurvatures and supertorsions, as we shall discuss below. We begin, however, with a rather elementary introduction intended for readers who are unfamiliar with supersymmetry and supergravity.

## 2 Basics of supergravity

As with any gauge theory, one can either approach supergravity by first studying its coupling to matter, or one can begin by constructing the gauge action. The gauge field for supersymmetry is the spin 3/2 field  $\psi_\mu$  which is called the gravitino field. It is clear that this gauge field should be a vector-spinor field of the form  $\psi_\mu^\alpha$  ( $\mu = 0, 3$  and  $\alpha = 1, 4$ ) because gauge fields transform as  $\delta(\text{gauge field}) = \partial_\mu(\text{parameter}) + \dots$ . For supersymmetry the parameter is a spinor  $\epsilon^\alpha$ , so the gauge field should have the index structure  $\psi_\mu^\alpha$ .

It is real (because it is the partner of the gravitational field, see below), and of course anticommuting as the spin-statistics relation suggests. As a vector-spinor, it contains on-shell only helicities  $\pm 3/2$ , but off-shell the field  $\psi_\mu^\alpha$  also contains spin  $1/2$  parts, just like a gauge field  $A_\mu$  contains on-shell helicities  $\pm 1$ , and off-shell also helicity zero.

In  $3 + 1$  dimensions, one can have theories with  $N = 1$  up to  $N = 8$  real gravitinos, but beyond  $N = 8$  the massless representations of the underlying supersymmetry algebra contain particles with spin larger than 2, and no consistent gauge theories exist for these cases.

For the simplest case,  $N = 1$ , the supersymmetry algebra has massless representations in terms of physical states with adjacent helicities  $(J, J + 1/2)$ . By combining these representations with representations with helicities  $(-J, -J - 1/2)$ , one obtains the field content for massless fields with spin  $J$  and  $J + 1/2$ . This is a result of Salam and Strathdee who also pioneered the superspace approach [3]. The gauge action for  $N = 1$  supersymmetry is based on the multiplet with  $(J, J + 1/2) = (\frac{3}{2}, 2)$ . The alternative, spin  $(1, 3/2)$ , does not lead to a consistent supergravity theory; it couples the Maxwell field to gravitinos but the resulting gauge theory contains no physical particles (the curvatures vanish according to the field equations of this model). However, one can view the spin  $(1, 3/2)$  multiplet as a matter multiplet, and couple it to the spin  $(3/2, 2)$  gauge multiplet. The result is the simplest extended supergravity theory, the  $N = 2$  model with spin content  $(1, 3/2, 3/2, 2)$ . It realizes Einstein's goal of unifying gravity with Maxwell electromagnetism, and has an  $O(2)$  symmetry which rotates the two gravitinos into each other [7]. In the same way the  $N = 8$  model has a local  $SO(8)$  symmetry [8]. The group  $SO(8)$  is not big enough to contain the nongravitational symmetry group  $SU(3) \times SU(2) \times U(1)$ , and this precluded direct contact with phenomenology.

Because supersymmetry requires that the gravitino is part of a spin  $(3/2, 2)$  multiplet, a gauge theory of local supersymmetry necessarily contains gravity. Thus gauge supersymmetry is a theory of gravity, and this explains the name supergravity. One could also arrive at this name by starting with a matter theory of rigid supersymmetry, and couple it to gravity. Because the supersymmetry parameter  $\epsilon^\alpha$  transforms as a spinor under local

Lorentz transformation, it becomes in general spacetime-dependent after a local Lorentz transformation, even if it initially was rigid (spacetime independent), hence also from this point of view the name supergravity seems appropriate.

In their pioneering article, Yang and Mills first wrote down kinematical transformation rules, and only afterwards constructed a gauge action for the group  $SU(2)$ . Before them, one usually began with an action with particular dynamics, and then set out to describe the symmetries of the dynamical model under consideration. Likewise in supergravity one can begin either by first constructing the gauge action, or the coupling to matter, or both, and afterwards discuss the symmetries, or one can begin with the symmetries, and afterwards construct actions with these symmetries. In supergravity there is a special problem with symmetries which does not occur in Yang-Mills theories, the problem of auxiliary fields. If one studies the transformation laws of the local symmetries, which are for the  $N = 1$  supergravity theory the local Lorentz and general coordinate symmetry of ordinary gravity and further local supersymmetry, then one discovers that “the local gauge algebra does not close” without auxiliary fields. Namely, the commutator of two local supersymmetry transformations is not only a sum of local symmetries, but one finds also field equations (of fermionic fields in general, but not always [9]) on the right-hand side [10]. Thus one does not have a representation of the local gauge algebra in terms of fields. In some cases (but not all cases) one can add a few “auxiliary fields” (fields whose field equations do not describe physical states) such that the local gauge algebra closes. In the early literature on supergravity, finding a set of auxiliary fields was an art.

Having a closed gauge algebra allows one to go to superspace. Superspace is a space with fermionic coordinates  $\theta^{i\alpha}$  (with  $i = 1, \dots, N$ ) in addition to the bosonic coordinates. In superspace one can define supercurvatures and supertorsions, but contrary to ordinary general relativity, in supergravity these supercurvatures and supertorsions must satisfy certain constraints. These constraints define the geometry; they are inserted by hand from the outside, and are not field equations. Again, in the early days finding a correct set of constraints was an art. (Correct means here: such that no ghosts and no particles with spin larger than 2 do occur). A translation method was constructed, called gauge

completion, which could map a supergravity theory which was given in  $x$ -space, into superspace [11], but this was only possible if the local gauge algebra was closed. For this reason the auxiliary fields, even though not physical, were of great importance. It should be mentioned that the superspace approach contains superfields (fields depending on  $x^\mu$  and  $\theta^{i\alpha}$ ) which contain many more local symmetries and many more auxiliary fields than the corresponding theory in  $x$ -space. The map from  $x$ -space into superspace corresponds to a particular gauge of the full superspace theory. However, working with non-gaugefixed superfields simplifies the calculations a good deal.

The classical supergravity theories are gauge field theories. To quantize them one followed initially the same procedures as followed by 't Hooft and Veltman and others for Yang-Mills gauge theories. Namely, one added a gauge-fixing term and corresponding Faddeev-Popov ghost action. For the gravitinos the most useful gauge fixing term was  $\mathcal{L}(\text{fix}) = \frac{1}{4\xi}(\bar{\psi}_\mu \gamma^\mu) \not{D}(\gamma^\nu \psi_\nu)$  with  $\xi = 1$  [12]. Because ghosts have opposite statistics from the corresponding local gauge parameters, and the parameters for supersymmetry are anticommuting (Grassmann variables), the ghosts for local supersymmetry were commuting spinors. In addition the ghosts for local Lorentz invariance  $c^{mn} = -c^{nm}$  and the ghosts for general coordinate transformations  $c^\mu$ , were anticommuting. The ghosts for local Lorentz symmetry could be eliminated by their algebraic field equations. However, a new feature in the quantization of gauge theories was discovered. Certain supergravity theories contain antisymmetric tensor fields (for example the supergravity theories in ten dimensions), and for these theories, the Faddeev-Popov ghost actions were themselves gauge actions! Thus one had to do the Faddeev-Popov procedure all over again, and in this way ghosts-for-ghosts emerged. To deal with this complicated issue a general framework was developed by Batalin and Vilkovisky, which generalized earlier work by Zinn-Justin, and which is nowadays called the antifield formalism [13]. The antifields in this approach are external fields which satisfy bracket relations with the original fields; it is a kind of covariant Hamiltonian formalism. However, the antifields have opposite statistics from the original field, and consequently the bracket itself is anticommuting.

The antifield formalism made contact with the BRST formalism which was soon es-

established after 't Hooft and Veltman had renormalized Yang-Mills theory. The authors (Becchi, Rouet, Stora, and later Tyutin [14]) noticed that after covariant quantization the quantum action (the sum of the classical action and the gauge fixing term and the ghost action) still had a residual rigid symmetry with a constant anticommuting parameter  $\Lambda$ . Combining the antifield formalism with BRST symmetry has led to a profound geometrical framework. It generalizes the work of Dirac, Heisenberg and Pauli, and Gupta and Bleuler on QED to the nonabelian level. In earlier studies of  $N = 1$  supergravity it had been found that unitarity required four-ghost couplings if one did not add auxiliary fields [15]. The BRST-antifield formalism shows that four-ghost couplings are a direct consequence of the nonclosure of the gauge algebra. On the other hand, if one starts with a classical formulation of supergravity with auxiliary fields and with closed gauge algebra, then standard Faddeev-Popov quantization is applicable (in the simplest models at least), and eliminating the auxiliary fields from the quantum action, the same four-ghost couplings are found. This demonstrates the close connection between gauge algebras, quantization and geometry.

### 3 Simple ( $N = 1$ ) supergravity.

The simplest way to introduce supergravity is to begin with ordinary gravity, add the action for a free spin 3/2 field, couple it to gravity according to the usual rules of general relativity, and then try to make the action invariant under local supersymmetry. We shall follow this procedure, but go back to Herman Weyl's 1929 formulation of gravity [16], and generalize it to supergravity.

According to this approach one begins with the spacetime symmetry algebra of the tangent frames, the Lorentz algebra with generators  $M_{mn}$  satisfying  $[M_{mn}, M_{pq}] = \eta_{np}M_{mq} + 3$  further terms. Then one associates a gauge field (connection) to  $M_{mn}$ , the spin connection  $\omega_\mu^{mn} = -\omega_\mu^{nm}$ . Finally one constructs the Yang-Mills curvature for the Lorentz group. In general a Yang-Mills curvature has the form

$$F_{\mu\nu}{}^a = \partial_\mu\omega_\nu{}^a - \partial_\nu\omega_\mu{}^a + f_{bc}^a\omega_\mu{}^b\omega_\nu{}^c \quad (1)$$

and in the case of the Lorentz group one obtains, using the structure constants of the Lorentz group,

$$R_{\mu\nu}{}^{mn}(\omega) = \partial_\mu \omega_\nu{}^{mn} - \partial_\nu \omega_\mu{}^{mn} + \omega_\mu{}^{mp} \eta_{pq} \omega_\nu{}^{qn} - \omega_\mu{}^{np} \eta_{pq} \omega_\nu{}^{qm} \quad (2)$$

Gauge transformations read in general

$$\delta W_\mu{}^a = \partial_\mu \lambda^a + g f_{bc}^a W_\mu{}^b \lambda^c \quad (3)$$

and become for the Lorentz group

$$\delta \omega_\mu{}^{mn} = D_\mu \lambda^{mn} \equiv \partial_\mu \lambda^{mn} + \omega_\mu{}^{mn} \eta_{pq} \lambda^{qn} + \omega_\mu{}^{np} \eta_{pq} \lambda^{mq} \quad (4)$$

The curvatures transform homogeneously,

$$\delta(\lambda) R_{\mu\nu}{}^{mn} = \lambda^m{}_p R_{\mu\nu}{}^{pn} + \lambda^n{}_p R_{\mu\nu}{}^{mp} \quad (5)$$

To construct an action Weyl was faced with the problem of contracting the indices of  $R_{\mu\nu}{}^{mn}$ . It was natural to consider an action linear in  $R_{\mu\nu}{}^{mn}$  because that was Einstein's approach (but an alternative is an  $R^2$  action, see below). To this purpose he considered the vielbein fields  $e_\mu{}^m$  which naturally arise if one tries to put the Dirac action in curved space. The Dirac matrices satisfy  $\{\gamma^m(x), \gamma^n(x)\} = 2\eta^{mn}$  in flat space, but in curved space  $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$  where  $g^{\mu\nu}(x)$  is the metric, and then it is natural to write  $\gamma^\mu(x) = \gamma^m e_m{}^\mu(x)$  [17]. Latin indices  $(m, n)$  correspond to tensors in flat space (the tangent frames, the freely falling lifts) while Greek indices  $(\mu, \nu)$  correspond to coordinates in curved space. Substituting the expression for  $\gamma^\mu(x)$  into the Clifford algebra immediately yields

$$\eta^{mn} e_m{}^\mu e_n{}^\nu = g^{\mu\nu} \quad (6)$$

Thus Weyl constructed the Einstein-Hilbert action for gravity in terms of the spin connection and vielbein fields

$$\mathcal{L} = -\frac{1}{2} e R_{\mu\nu}{}^{mn}(\omega) e_m{}^\nu e_n{}^\mu \quad (7)$$

where  $e = \det e_\mu{}^m$ , and  $e_m{}^\nu$  is the matrix inverse of  $e_\nu{}^m$ . Straightforward algebra shows that it is equal to the action Einstein and Hilbert had written down in terms of metrics



$g_{\mu\nu}$  and Christoffel connections  $\Gamma_{\mu\nu}^{\rho}(x)$  [18]. However, for the couplings to fermions, one needs Weyl's formulation.

For the extension of Weyl's approach to supergravity it is useful to consider the Poincaré algebra instead of the Lorentz algebra with  $[M_{mn}, P_l] = \eta_{ml}P_m - \eta_{ml}P_n$  and  $[P_m, P_n] = 0$ . We associate again  $\omega_{\mu}^{mn}$  with  $M_{mn}$ , but  $e_{\mu}^m$  can now be associated with  $P_m$ . For our purposes we must generalize this to an anti-de Sitter algebra, where instead of  $[P_m, P_n] = 0$  one has  $[P_m, P_n] = \alpha^2 M_{mn}$  with  $\alpha$  a free parameter. The Yang-Mills curvatures now become

$$R_{\mu\nu}^{mn}(\omega, e) = R_{\mu\nu}^{mn}(\omega) + \alpha^2(e_{\mu}^m e_{\nu}^n - e_{\mu}^n e_{\nu}^m) \quad (8)$$

$$R_{\mu\nu}^m(\omega, e) = \partial_{\mu}e_{\nu}^m - \partial_{\nu}e_{\mu}^m + \omega_{\mu}^{mp}\eta_{pq}e_{\nu}^q - \omega_{\nu}^{mp}\eta_{pq}e_{\mu}^q \quad (9)$$

In order to generalize the Poincaré algebra to a superalgebra which can be used for supergravity, one needs anticommuting generators. The spin-statistics connection suggests that these parameters should be spinors. Spinors can be described in a four-component formalism or in a two-component formalism<sup>2</sup>. Although two-component spinors are widely used, and form the irreducible representations of the Lorentz group, we shall first use four-component spinors to reach a wider audience. We consider generators  $Q^{\alpha}$  ( $\alpha = 1, \dots, 4$ ). They transform as spinors under the Lorentz group,  $[M_{mn}, Q^{\alpha}] = \frac{1}{4}([\gamma_m, \gamma_n])^{\alpha}_{\beta} Q^{\beta}$ , and they are constant in space and time,  $[Q^{\alpha}, P_m] = 0$ . If supersymmetry is to map bosons into fermions, and fermions into bosons, there should be no kernel for  $Q^{\alpha}$  (the null space of  $Q^{\alpha}$  should be trivial). Hence  $\{Q^{\alpha}, Q^{\beta}\}$  should be equal to an operator which has no kernel. Covariance and the fact that the only commuting generators available are  $P_m$  and  $M_{mn}$ , allows then only

$$\{Q^{\alpha}, Q^{\beta}\} = \gamma^{m, \alpha\beta} P_m + \alpha' (\gamma^{[m} \gamma^{n]})^{\alpha\beta} M_{mn} \quad (10)$$

In fact, for the superPoincaré algebra  $\alpha' = 0$ , but for the super-anti de Sitter algebra  $\alpha'$  is equal to the parameter  $\alpha$  we already encountered. Consistency (satisfying the Ja-

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<sup>2</sup>Soon after a tensor calculus was established for general relativity, Ehrenfest in Leiden sent a letter to van der Waerden in Göttingen, asking if something similar could be done for spinors. These result is “the van der Waerden formalism” of two-component dotted and undotted spinor indices  $A, \dot{A}$  [19].

cobi identities) then also requires  $[Q^\alpha, P_m] = \alpha(\gamma_m)^\alpha_\beta Q^\beta$ . The corresponding Yang-Mills curvatures are

$$R(M) = R_{\mu\nu}{}^{mn}(\omega, e, \psi) = R_{\mu\nu}{}^{mn}(\omega) - \alpha \bar{\psi}_\mu \gamma^{[m} \gamma^{n]} \psi_\nu + \alpha^2 (e_\mu{}^m e_\nu{}^n - e_\mu{}^n e_\nu{}^m) \quad (11)$$

$$R(P) = R_{\mu\nu}{}^m(\omega, e, \psi) = R_{\mu\nu}{}^m(\omega, e) + \bar{\psi}_\mu \gamma^m \psi_\nu \quad (12)$$

$$R(Q) = R_{\mu\nu}{}^\alpha(\omega, e, \psi) = \left( D_\mu(\omega) \psi_\nu + \frac{1}{2} \alpha e_\mu{}^m \gamma_m \psi_\nu \right) - \mu \rightarrow \nu \quad (13)$$

The gauging of superalgebras has been discussed in [20].<sup>3</sup>

Minkowski space can be viewed as a coset space, namely Poincaré algebra/Lorentz algebra, and anti-de Sitter space is the coset space  $SO(3, 2)/SO(3, 1)$ . Superspace is in the same way the coset space  $\{M_{mn}, P_m, Q^\alpha\}/\{M_{mn}\}$ , namely super-Poincaré algebra/Lorentz algebra, or super-anti de Sitter algebra/Lorentz algebra. The curvatures of the coset generators are usually called torsions. According to this terminology, the Lorentz curvature is a genuine curvature, but  $R_{\mu\nu}{}^m$  and  $R_{\mu\nu}{}^\alpha$  are torsions.

Having come so far, it is natural to follow Yang and Mills and construct an action for supergravity, quadratic in curvatures, and invariant under the two local spacetime symmetries (local supersymmetry will be discussed later). One can still contract the indices of the curvatures in various ways with vielbein fields. But there is one way which uses constant tensors, just as Yang and Mills used in their paper, and that is by using  $\epsilon$ -tensors [21]

$$\mathcal{L} = R_{\mu\nu}{}^{mn} R_{\rho\sigma}{}^{pq} \epsilon^{\mu\nu\rho\sigma} \epsilon_{mnpq} + a (\bar{R}_{\mu\nu})_\alpha (\gamma_5)^\alpha_\beta (R_{\rho\sigma})^\beta \epsilon^{\mu\nu\rho\sigma} \quad (14)$$

where  $(\bar{R}_{\mu\nu})_\alpha = (R_{\mu\nu}{}^\beta)^\dagger i (\gamma^0)^\beta_\alpha$  with  $(\gamma^0)^2 = -1$ , and  $(\gamma_5)^2 = +1$ , while  $a$  is a constant to be fixed later. Note that

(i) parity is preserved

(ii) the action is a density (because the tensor  $\epsilon^{\mu\nu\rho\sigma}$  with entries  $\pm 1, 0$  is a density)

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<sup>3</sup>To check signs and the Jacobi identities, an easy method is to assume that the curvature two-forms vanish, and then to check that the exterior derivative of them also vanishes. Note that the two forms  $\bar{\psi}\psi$  and  $\bar{\psi}\gamma^{mnp}\psi$  vanish for Majorana spinors.

(iii) no term with the square of  $R_{\mu\nu}{}^m$  can be constructed in this way.

(iv) all fields have “geometrical dimensions”, meaning that the gravitational coupling constant has been absorbed into the gravitino. So there will be no gravitational coupling constants in any of the formulas below, but note that  $\alpha$  and  $a$  are dimensional. The gravitino has dimension 1/2, and  $\alpha$  and  $a$  have dimension 1.

Now comes a surprise: substituting the expression for  $R_{\mu\nu}{}^{mn}(\omega, e, \psi)$  into  $\mathcal{L}$ , the leading term

$$\epsilon^{\mu\nu\rho\sigma}\epsilon_{mnpq}R_{\mu\nu}{}^{mn}(\omega)R_{\rho\sigma}{}^{pq}(\omega) \quad (15)$$

is a total derivative [21]. (In form language it reads  $R^{mn}(\omega) \wedge R^{pq}(\omega)\epsilon_{mnpq}$ . Under a variation  $\omega \rightarrow \omega + \delta\omega$  with arbitrary  $\delta\omega$ , one finds  $\delta R^{mn} = D\delta\omega^{mn}$  and then partial integration yields a vanishing result due to the Bianchi identity,  $DR = 0$ . Equivalently, locally  $d(R \wedge R) = DR \wedge R + R \wedge DR = 0$ , hence  $R \wedge R$  is locally a total derivative. These formulas hold for any spin connection  $\omega$ , whether it is an independent field or a dependent field). The cross terms in the bosonic sector yield the Einstein-Hilbert action<sup>4</sup> but in the formulation of Weyl

$$R_{\mu\nu}{}^{mn}(\omega)e_{\rho}{}^pe_{\sigma}{}^q\epsilon_{mnpq}\epsilon^{\mu\nu\rho\sigma} \sim eR(e, \omega) \quad (16)$$

while the four vielbein fields yields a cosmological constant

$$(e_{\mu}{}^me_{\nu}{}^ne_{\rho}{}^pe_{\sigma}{}^q)\epsilon^{\mu\nu\rho\sigma}\epsilon_{mnpq} \sim e \quad (17)$$

In the fermionic sector one now finds similar results [21]:

- the leading term  $(D_{\mu}\bar{\psi}_{\nu}\gamma_5 D_{\rho}\psi_{\sigma})\epsilon^{\mu\nu\rho\sigma}$  cancels the cross term  $\alpha R(M)\bar{\psi}\gamma_{mn}\psi$  in  $R(M)R(M)$  up to a total derivative.
- the cross terms in  $R(Q)R(Q)$  yield the gauge action for the gravitinos

$$\mathcal{L}_{3/2} = (\bar{\psi}_{\mu}\gamma_{\nu})\gamma_5(D_{\rho}\psi_{\sigma})\epsilon^{\mu\nu\rho\sigma} \sim \bar{\psi}_{\mu}\gamma^{\mu\rho\sigma}D_{\rho}\psi_{\sigma} \quad (18)$$

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<sup>4</sup>Use that  $\epsilon_{mnpq}\epsilon^{\mu\nu\rho\sigma} = e_m{}^{\mu}e_n{}^{\nu}e_p{}^{\rho}e_q{}^{\sigma} + 23$  other terms, due to antisymmetrization in  $\mu, \nu, \rho, \sigma$ .

(by  $\gamma^{\mu\rho\sigma}$  we mean  $\gamma^\mu\gamma^\rho\gamma^\sigma$  antisymmetrized in  $\mu, \rho, \sigma$ ). This is the action Rarita and Schwinger first wrote down (up to a field redefinition as in (35)) when they studied nuclear beta decay under the assumption that neutrinos have spin 3/2 [22].

- the remaining term is the masslike term  $(\bar{\psi}_\mu\gamma_\nu\gamma_5\gamma_\rho\psi_\sigma)\epsilon^{\mu\nu\rho\sigma} \sim \bar{\psi}_\mu\gamma^{\mu\nu}\psi_\nu$ , which is needed in the presence of a cosmological constant in order that local supersymmetry is preserved so that the gravitino remains massless [23].

The action is manifestly invariant under local Lorentz and Einstein transformations (general coordinate transformations). What can now be said about the local supersymmetry of this action? The Yang-Mills transformation rules for local supersymmetry follow directly from the super anti-de Sitter algebra. Applying the general rules of Yang and Mills but with the structure constants of the super anti-de Sitter algebra yields<sup>5</sup>

$$\delta e_\mu^m = -\bar{\epsilon}\gamma^m\psi_\mu \quad (19)$$

$$\delta\psi_\mu = \partial_\mu\epsilon - \frac{\alpha}{2}e_\mu^m\gamma_m\epsilon + \frac{1}{4}\omega_\mu^{mn}\gamma_{mn}\epsilon \quad (20)$$

$$\delta\omega_\mu^{mn} = \alpha\bar{\epsilon}\gamma^{mn}\psi_\mu \quad (21)$$

The action is **not** invariant under these transformation rules. The reason is, of course, that the curvatures in (14) were contracted with constants which are not invariant tensors under local supersymmetry. At this point a new subtlety arises which is absent in Yang-Mills theories: constraints are needed on the curvatures to obtain the correct law for  $\delta\omega_\mu^{mn}$ .

Variation of the action under local supersymmetry transformations (an easy task since curvatures rotate into curvatures<sup>6</sup>) leads to  $R(Q)R(M)$  terms which cancel if  $a = 8\alpha$ . However, a term proportional to  $R(P)R(Q)$  is left. Since  $R(P) = 0$  in (12) can be algebraically solved (see below) but  $R(Q) = 0$  in (13) can not be algebraically solved, we

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<sup>5</sup>Again a simple way to check these results is to write the transformation rules as one-forms and to verify that the curvature two-forms transform into each other.

<sup>6</sup>The transformation rules of the curvatures are obtained by replacing the connections by their corresponding curvatures in (19-21).

impose the torsion constraint

$$R_{\mu\nu}{}^m(P) = 0. \quad (22)$$

Its solution yields  $\omega_\mu{}^{mn}$  as a function of  $e_\mu{}^m$  and  $\psi_\mu$ ,

$$\omega_\mu{}^{mn} = \omega_\mu{}^{mn}(e) - \frac{1}{2}(\bar{\psi}_\mu \gamma^m \psi^n + \bar{\psi}^m \gamma_\mu \psi^n - \bar{\psi}_\mu \gamma^n \psi^m) \quad (23)$$

where  $\omega_\mu{}^{mn}(e)$  is the spin connection in terms of vielbein fields which one can find in text books on general relativity. The terms with  $\bar{\psi}\gamma\psi$  are torsion. Torsion was introduced into general relativity in the 1920's by E. Cartan [24], but it has found its natural realization in supergravity. The constraint in (22) is also the field equation of the spin connection in  $N = 1$   $x$ -space supergravity, see (26), but in superspace or conformal supergravity the constraints are not field equations. One can now determine  $\delta\omega_\mu{}^{mn}(e, \psi)$  by applying the chain rule and using (19) and (20). Then one can check the invariance of the action by varying all fields. This is laborious (and the way it was first done); in particular, the variation of  $\omega(e, \psi)$  leads to a lot of terms if one uses the chain rule. The crucial observation, arrived at much later, is that the  $\omega$  field equation (whose solution is  $\omega = \omega(e, \psi)$  with a complicated  $\omega(e, \psi)$ ) is identically satisfied, once one uses everywhere  $\omega(e, \psi)$  instead of  $\omega$ . The reason is that after substituting  $\omega(e, \psi)$  for  $\omega$ , the variation of the composite object  $\omega(e, \psi)$  is always multiplied by  $\delta S/\delta\omega$  which is zero. Hence, one can forget about the variation of  $\omega(e, \psi)$  altogether, provided one works in second-order formalism with a dependent field  $\omega$ .

It is amusing to see how elegantly this all works out. From  $R(P) = 0$  one finds that the extra variation  $\Delta\omega^m{}_n$  is determined by

$$\delta R(P)^m = -\bar{\epsilon}\gamma^m R(Q) + \Delta\omega^m{}_n e^n = 0 \quad (24)$$

From this expression one can solve for  $\Delta\omega^m{}_n$ . The total variation of the spin connection is then  $\delta\omega^m{}_n + \Delta\omega^m{}_n$ , and this expression agrees with the result of applying the chain rule to (23) and using (19) and (20).

It is also easy to see that the sum of all variations due to  $\delta\omega^m{}_n + \Delta\omega^m{}_n$  cancels. Variation of the action, and partial integration of the Einstein-Hilbert term yields the

following result in form notation

$$\delta\mathcal{L} = 2(\delta\omega^{mn} + \Delta\omega^{mn})[\epsilon_{mnpq}DR(M)^{pq} + a\bar{R}(Q)\gamma_5\frac{1}{4}\gamma_{mn}\psi] \quad (25)$$

Using  $DR(\omega) = 0$  and  $R(P) = 0$ , the terms inside the square bracket cancel provided again  $a = 8\alpha$ .

Originally, Freedman, Ferrara and the author began with  $\omega(e)$ , and found successive terms in the action and transformation laws by computer until they obtained an invariant action [1]. Two thousand terms had to cancel, and did cancel. Looking back, it is now clear that these extra terms just replaced  $\omega(e)$  by  $\omega = \omega(e, \psi)$ . Clearly then, imposing the constraint  $R(P) = 0$ , which replaces  $\omega$  by  $\omega(e, \psi)$  at the beginning, is an enormous simplification.

One can also work with an independent field  $\omega$ . This is called first-order formalism. Then one has no constraint  $R(P) = 0$ , and one must find  $\delta(\epsilon)\omega$  by direct means. This can be done, and was done by Deser and Zumino [2]. Also this result is complicated, and not equal to the transformation laws found in the second order formalism.

In fact, one can even more clearly show where the choice between first-order and second-order formalism is made. The variation of the action with respect to  $\omega$  can be written as

$$\delta S \sim \int \epsilon^{\mu\nu\rho\sigma} \epsilon_{mnpq} R_{\mu\nu}{}^m(P) [\hat{\delta}\omega_\rho{}^{nr} - \Omega_\rho{}^{nr}(e, \psi)] e_\sigma{}^s \quad (26)$$

where  $\Omega_\rho{}^{nr}(e, \psi)$  is a complicated function and  $\hat{\delta}\omega_\rho{}^{nr}$  denotes any variation  $\omega_\rho{}^{nr}$ . We only used  $\delta e_\mu{}^m = -\bar{\epsilon}\gamma^m\psi_\mu$  and  $\delta\psi_\mu = D_\mu(\omega)\epsilon$  to obtain this result. The coefficient of  $\hat{\delta}\omega$  is, of course, the  $\omega$  field equation, according to the general Euler-Lagrange variational principle. The fact that  $\delta S$  factorizes is nontrivial and leads to the choice between first- and second-order formalism. Second-order formalism puts  $R_{\mu\nu}{}^m(P) = 0$ , thus replacing the independent field  $\omega_\mu{}^{mn}$  by the dependent field  $\omega_\mu{}^{mn}(e, \psi)$ . This is the result of [1]. First-order formalism puts  $\delta\omega$  equal to  $\Omega$  and this yields the result of [2].

In an early article, Volkov and Soroka [25] gauged the super Poincaré algebra (not the super anti de Sitter algebra), but did not prove its invariance under supersymmetry. They used a first-order formalism and found  $\delta\omega = 0$ . This agrees with (21) in the limit

$\alpha = 0$ . As we have explained, this result is incorrect. They implicitly assumed that their action would be invariant, and concluded that supergravities exist for any  $N$ . Careful study, using  $\delta\omega \neq 0$ , shows that  $N \leq 8$ .

The fact that in second-order formalism one need not take into account the variations of  $\omega = \omega(e, \psi)$  is sometimes called 1.5-order formalism because it combines in some sense the virtues of first- and second-order formalisms. Namely, one keeps the composite object  $\omega = \omega(e, \psi)$  as one object (not expanding it in terms of  $e$  and  $\psi$ ), and this is like first-order formalism. But then one uses that  $\delta\omega(e, \psi)$  is multiplied by  $\delta S/\delta\omega$  which is identically zero, and this is due to using second-order formalism. So, in the end only  $\delta e_\mu^m = -\bar{\epsilon}\gamma^m\psi_\mu$  and  $\delta\psi_\mu = D_\mu\epsilon$  are needed. This makes the proof of the supersymmetry invariance of the action for  $N = 1$  supergravity very easy, as easy as the gauge invariance of the action of Yang and Mills.

## 4 Covariant quantization.

Next we briefly discuss the covariant quantization of supergravity. We consider a simplified case, with only external gravitational fields and no Einstein action. To preserve covariance at the quantum level, one may use a background field formalism. The spin 3/2 action has then a torsionless spin connection

$$\mathcal{L}_{3/2} = -\frac{1}{2}e\bar{\psi}_\mu\gamma^{\mu\rho\sigma}D_\rho\psi_\sigma \quad (27)$$

$$D_\rho\psi_\sigma = \partial_\rho\psi_\sigma + \frac{1}{4}\omega_{\rho mn}(e)\gamma^m\gamma^n\psi_\sigma \quad (28)$$

It is invariant by itself under local supersymmetry transformations, without adding the Einstein action, if the gravitational background fields are Ricci flat ( $R_{\mu\nu} = 0$ ). We use a supersymmetry gauge fixing term which preserves the classical spacetime symmetries

$$\mathcal{L}(\text{fix}) = \frac{1}{4}e\bar{\psi}_\mu\gamma^\mu\mathcal{D}\gamma^\nu\psi_\nu \quad (29)$$

The Faddeev-Popov action for the supersymmetry ghosts is

$$\mathcal{L}(\text{ghost}) = -e\bar{b}_\alpha\mathcal{D}c^\alpha \quad (30)$$

where  $b^\alpha$  and  $c^\alpha$  are 4-component commuting Majorana spinors (or, more precisely,  $c^\alpha$  is a Majorana spinor but  $b^\alpha$  is  $i$  times a Majorana spinor). To obtain  $\mathcal{L}(\text{fix})$  in the action, one starts from a gauge fixing term  $\delta[\gamma^\mu\psi_\mu - F]$  in the path integral, and then one inserts unity into the path integral as follows

$$I = \int dF e^{\bar{F}\mathcal{D}F} (\det \mathcal{D})^{-1/2} \quad (31)$$

Integration over  $F$  brings  $\mathcal{L}(\text{fix})$  in the action, but exponentiating  $(\det \mathcal{D})^{-1/2}$  with Nielsen-Kallosh ghosts [26] yields

$$\mathcal{L}(\text{NK}) = -\frac{e}{2}\bar{A}\mathcal{D}A - \frac{e}{2}\bar{B}\mathcal{D}C \quad (32)$$

where  $A$  is a Majorana anticommuting ghost and  $B$  and  $C$  are Majorana commuting ghosts. The Faddeev-Popov ghosts remove the unphysical longitudinal and timelike parts of  $\psi_\mu$ , which correspond to the gauge symmetry  $\delta\psi_\mu = \partial_\mu\epsilon$  as in QED. The Nielsen-Kallosh ghosts, on the other hand, remove the spin 1/2 parts  $\gamma \cdot \psi$  from the spectrum.

The sum of the classical spin 3/2 action and the gauge fixing term leads to

$$\mathcal{L}_{3/2} = \frac{e}{4}\bar{\psi}_\mu\gamma^\sigma\mathcal{D}\gamma^\mu\psi_\sigma \quad (33)$$

This is a complicated action. One can, however, reduce it to the Dirac action by some simple field redefinitions. We do this in  $n$  dimensions. We use in  $n$  dimensions as gauge fixing term

$$\mathcal{L}(\text{fix}) = \frac{n-2}{8}e\bar{\psi}_\mu\gamma^\mu\mathcal{D}\gamma^\nu\psi_\nu \quad (34)$$

This term vanishes for  $n = 2$ , as does the classical action in (27). We then choose a new basis for the spin 3/2 field

$$\chi_\mu = \psi_\mu - \frac{1}{2}\gamma_\mu\gamma \cdot \psi, \quad \psi_\mu = \chi_\mu - \frac{1}{n-2}\gamma_\mu\gamma \cdot \chi \quad (35)$$

One finds that the action in  $n$  dimensions on the basis  $\chi_\mu$  becomes a sum of Dirac actions

$$\mathcal{L}_{3/2} + \mathcal{L}(\text{fix}) = -\frac{e}{2}\bar{\chi}_\mu\mathcal{D}\chi^\mu = -\frac{1}{2}e\bar{\chi}_m\mathcal{D}\chi^m \quad (36)$$

where  $\chi_m = e_m{}^\mu\chi_\mu$ . The operator  $\mathcal{D}$  contains now both a term with the spin connection which acts on the spinor index, and another term which acts on the vector index of  $\chi^m$ . This form of the spin 3/2 action has been used to compute chiral anomalies using quantum mechanics [27].



## 5 Conformal simple supergravity.

We now apply the formalism to a second, far less trivial, example: conformal simple supergravity [28]. Simple means again that there is one ordinary supersymmetry generator  $Q_\alpha$  ( $\alpha = 1, 4$ ), but there is also a conformal-supersymmetry generator  $S_\alpha$ . (There also exist extended conformal supergravities, but only with  $1 \leq N \leq 4$ , unlike ordinary supergravities for which  $1 \leq N \leq 8$ .) The bosonic conformal algebra contains the translation generators  $P_m$  ( $m = 0, 3$ ) which now commute with each other and with  $Q_\alpha$ , but there are also conformal boost generators  $K_m$ , Lorentz generators  $M_{mn}$  and scale generators  $D$ . The pair  $P_m, Q_\alpha$  resembles the pair  $K_m, S_\alpha$ ; for example, also  $[K_m, K_n] = 0$  and  $[K_m, S_\alpha] = 0$ . One might expect that the total set of generators consist of  $\{P_m, M_{mn}, D, K_m\}$  and  $\{Q_\alpha, S_\alpha\}$ . However, in the superalgebra one needs one more bosonic generator  $A$  for chiral transformations of  $Q_\alpha$  and  $S_\alpha$ . In the literature the corresponding symmetry is called  $R$  symmetry.

Conformal symmetry is believed to be the symmetry of fundamental interactions at ultra-high energies where masses can be neglected. It is a larger symmetry than only scale invariance, and whether at some deep level Nature has conformal invariance is an open question. According to the criteria of perturbative quantum field theory, conformal gravity is not viable as a low-energy effective action because its propagator has double poles and violates unitarity [29]. However, gravity theories are inherently nonperturbative, so maybe this is not the whole story. In string theories conformal, or rather superconformal, symmetry plays a big role, but we shall here not go further into the physics of conformal symmetry. Rather, we want to construct a gauge action, hoping that one day it will be used.

All generators have a  $Z$  grading: if two generators have grades  $p$  and  $q$ , their commutator contains only generators with grade  $p + q$ . The generators together with their grades are as follows:

$$\left\{ \begin{array}{ccccc} & & D, A & & \\ K_m & S_\alpha & M_{mn} & Q_\alpha & P_m \\ -2 & -1 & 0 & 1 & 2. \end{array} \right. \quad (37)$$

This explains why  $[Q_\alpha, P_n] = 0$  and  $[P_m, P_n] = 0$  (idem for  $K_m, S_\alpha$ ). In addition,  $A$  commutes with the bosonic conformal algebra. We shall not write the superconformal algebra down in detail. It corresponds to  $SU(2, 2|1)$ , which may be defined by  $5 \times 5$  matrices with in the  $4 \times 4$  left-upper part the bosonic  $SU(2, 2) \sim SO(4, 2)$  conformal algebra, and further the generator  $A$  is represented by a diagonal matrix (with entries  $(1, 1, 1, 1, +4)$  up to an overall scale). In the fifth row and fifth column one finds  $Q_\alpha$  and  $S_\alpha$ . All matrices are supertraceless,  $\text{str} M = \sum_{i=1}^4 M^i_i - M^5_5 = 0$ .

To gauge this algebra, we introduce again for each generator a gauge field and a local gauge parameter. These gauge fields and parameters we denote by

$$f_\mu^m(\xi_K^m), \varphi_\mu^\alpha(\epsilon_S^\alpha), b_\mu(\lambda_D), A_\mu(\lambda_A), \omega_\mu^{mn}(\lambda^{mn}), \psi_\mu^\alpha(\epsilon_Q^\alpha), e_\mu^m(\xi_P^m). \quad (38)$$

The corresponding curvatures are denoted by

$$R_{\mu\nu}^m(K), R_{\mu\nu}^\alpha(S), R_{\mu\nu}(D), R_{\mu\nu}(A), R_{\mu\nu}^{mn}(M), R_{\mu\nu}^\alpha(Q), R_{\mu\nu}^m(P). \quad (39)$$

We must now construct an action quadratic in curvatures which preserves parity. In the vein of Yang and Mills we allow only constants to contract indices, but no vielbein fields. The action should be invariant under **all** local symmetries except  $P$ -gauge transformations, and under Einstein transformations. We expect again that we shall need constraints on the curvatures to achieve this, and we shall in a methodical way deduce these constraints, and solve them. In superspace one also finds constraints on curvatures and torsions.

Invariance under local scale transformations allows only products with curvatures with opposite grade, since the grades are proportional to the scale ( $D$  eigenvalue). Hence, we only consider the products  $R(K)R(P)$ ,  $R(S)R(Q)$ , and bilinears in  $R(M)$ ,  $R(D)$ ,  $R(A)$ . Since the chiral weights of  $R(Q)$  and  $R(S)$  are opposite,  $([A, Q_\alpha] = +c(\gamma_5)_\alpha^\beta Q_\beta$  while

$[A, S_\alpha] = -c(\gamma_5)_\alpha^\beta S_\beta$  with  $c$  a constant, local chiral invariance is then also achieved. Local Lorentz invariance requires to contract the curvatures with Lorentz-invariant tensors such as  $\epsilon^{\mu\nu\rho\sigma}$ ,  $(\gamma_m)^\alpha_\beta$ ,  $(\gamma_5)^\alpha_\beta$ . This will get us  $D, A, M$  invariance of the action. Parity restricts the coupling of  $R(A)$  to only  $R(D)$  (because  $A$  has negative parity, and  $D, M$  have positive parity). The most general action then reads

$$S = \int d^4x \epsilon^{\mu\nu\rho\sigma} [R_{\mu\nu}{}^{mn}(M) \epsilon_{mnr s} R_{\rho\sigma}{}^{rs}(M) + \alpha R(Q)_{\mu\nu}^\alpha (\gamma_5)_{\alpha\beta} R(S)_{\rho\sigma}^\beta + \beta R_{\mu\nu}(A) R_{\rho\sigma}(D)] \quad (40)$$

with  $\alpha, \beta$  constants. No  $R(P)R(K)$  coupling is possible which preserves parity. This is analogous to the observation that no term quadratic in  $R(P)$  was allowed in ordinary supergravity.

Before going on, we mention that the only curvatures which can lead to constraints which can be algebraically solved are  $R(P), R(Q), R(M)$  and  $R(D)$ . They contain terms with products of a vielbein  $e_\mu^m$  times another gauge field. Since we can invert the vielbein, we can eliminate the other gauge field. So ahead of time we know that our formalism should only lead to constraints on these curvatures.

We now study whether the action in (40) is invariant under the symmetries of the superconformal algebra using as transformation laws those of Yang and Mills, but with the structure constants of the superconformal algebra. We recall that the variation of the action only involves curvatures, as curvatures rotate into curvatures. We begin with the symmetries with negative grade. Invariance under local  $K$  and  $S$  transformations leads to two constraints on the curvatures and fixes the free parameters in the action

$$R_{\mu\nu}{}^m(P) = 0, \quad R_{\mu\nu}{}^\alpha(Q) \sim \epsilon^{\mu\nu\rho\sigma} R_{\rho\sigma}(Q)^\beta (\gamma_5)_\beta^\alpha, \quad \alpha = 8, \quad \beta = -4i. \quad (41)$$

This is indeed the complete action, but we need further constraints. In a conformally invariant action (in fact, scale invariance is enough for this argument) the fields with positive scale weight cannot appear in the kinetic part of the action, as there are no dimensionful constants which can make the action scale invariant. This explains ahead of time that  $f_\mu^m$ ,  $\varphi_\mu^\alpha$  must be eliminated by constraints. It comes perhaps not as a surprise that also  $\omega_\mu^{mn}$  will be eliminated just as in ordinary supergravity. Hence, at the end

the fields  $f_\mu{}^m$  (conformal vielbein),  $\varphi_\mu{}^\alpha$  (conformal gravitino) and  $\omega_\mu{}^{mn}$  (spin connection) will completely have been eliminated, leaving only  $e_\mu{}^m$  (graviton),  $\psi_\mu{}^\alpha$  (gravitino) and  $A_\mu$  (chiral gauge field) as physical fields. A special role will be played by  $b_\mu$ ; conformal symmetry (with local parameter  $\xi_K{}^m$ ) acts on  $b_\mu$  like  $\delta b_\mu = \xi_K{}^m \eta_{mn} e_\mu{}^n$ , and hence  $b_\mu$  can be gauged away by a suitable  $K$ -gauge choice. Equivalently: in the action  $b_\mu$  cancels.

So now we more or less know what we should find, and we shall now derive these results in a deductive manner, by studying the algebra (kinematics) rather than the action (dynamics). The first problem to face is that if there are constraints and one solves them by expressing one field in terms of others, this dependent field no longer transforms according to the (structure constants of the) superalgebra, but rather according to a result which follows from the chain rule. We already saw this in the case of the spin connection in simple ordinary supergravity. All constraints are invariant under all local symmetry transformations as given by the superalgebra, except under local  $P$  and local  $Q$  transformations. Thus, even after imposing and solving the constraints all dependent fields still transform as before imposing the constraints under all local symmetries except  $P$  and  $Q$ . This proves that the action is invariant under local  $K$ ,  $S$ ,  $M$ ,  $D$ ,  $A$  transformations. We shall replace the requirement that the action be invariant under local  $P$  transformations by the requirement that it be invariant under general coordinate (Einstein) transformations. Requiring invariance both under  $P$  and under Einstein transformations would be too much; however there is a deep relation between both as we now explain.

There exists a systematic procedure to find the constraints themselves. The basic idea is the observation that a general coordinate transformation differs from a gauge transformation by a curvature term

$$\begin{aligned}
\delta(\xi^\nu)\omega_\mu{}^a &\equiv \partial_\mu \xi^\nu \omega_\nu{}^a + \xi^\nu \partial_\nu \omega_\mu{}^a \\
&= \partial_\mu (\xi \cdot \omega^a) + f_{bc}^a \omega_\mu{}^b \xi \cdot \omega^c - \xi^\nu (\partial_\mu \omega_\nu{}^a - \partial_\nu \omega_\mu{}^a + f_{bc}^a \omega_\mu{}^b \omega_\nu{}^c) \\
&= D_\mu (\xi \cdot \omega^a) - \xi^\nu R_{\mu\nu}{}^a
\end{aligned} \tag{42}$$

We require the theory to be Einstein invariant, as well as gauge invariant under all symmetries *except*  $P$ . (We do not require invariance both under Einstein and under  $P$ -gauge

symmetries in order not to overcount.) If we consider now the commutator of two gauge transformations other than  $P$ , and produce a  $P$ -gauge transformation on the right hand side, we still have a symmetry if the difference of this  $P$  gauge transformation and an Einstein transformation vanishes. This difference is a curvature according to (42), and in this way one deduces the constraints on curvatures.

The only commutators which produce  $P$  generators on the right-hand side, but which do not involve  $P$  on the left-hand side, are (see the grade table) only the  $\{Q, Q\}$  anticommutator. So we shall compute  $\{Q, Q\}\omega_\mu^a$  taking for  $\omega_\mu^a$  first the gauge field with highest grade ( $e_\mu^m$ ), then the next ( $\psi_\mu^\alpha$ ) etc.

We begin with

$$[\delta(\epsilon_1^Q), \delta(\epsilon_2^Q)]e_\mu^m = \delta_P \left( \frac{1}{2} \bar{\epsilon}_2^Q \gamma^m \epsilon_1^Q \right) e_\mu^m. \quad (43)$$

Since  $\delta(\epsilon^Q)e_\mu^m \sim \bar{\epsilon}\gamma^m\psi_\mu$  and  $e_\mu^m$  and  $\psi_\mu^\alpha$  are physical fields, there will be no corrections:  $\Delta(\epsilon_Q)e_\mu^m = 0$  and  $\Delta(\epsilon_Q)\psi_\mu^\alpha = 0$ . On the other hand, we need a term  $(1/2)\bar{\epsilon}_2^Q \gamma^\lambda \epsilon_1^Q R_{\lambda\mu}^m(P)$  on the right-hand side to convert  $\delta_P$  into an Einstein transformation. This leads to the first constraint

$$R_{\mu\nu}^m(P) = 0. \quad (44)$$

It can be solved to give  $\omega_\mu^{mn} = \omega_\mu^{mn}(e, \psi, b)$ . (The field  $b_\mu$  cancels in the action, but not in  $\omega_\mu^{mn}$ .) This analysis is the same as in the previous section. In particular we find in the variation of  $\omega(e, \psi, b)$  a correction term  $\Delta\omega$ . Since  $R(P)$  rotates into  $R(Q)$ , the correction term  $\Delta\omega$  is proportional to  $R(Q)$ .

Next we consider the supersymmetry commutator on  $\psi_\mu^\alpha$ . Since there is an extra  $\Delta\omega_\mu^{mn}$ , one gets

$$[\delta(\epsilon_1^Q), \delta(\epsilon_2^Q)]\psi_\mu^\alpha = \delta_P \left( \frac{1}{2} \bar{\epsilon}_2^Q \gamma^m \epsilon_1^Q \right) \psi_\mu^\alpha - \left\{ \left( \frac{1}{4} \gamma_{mn} \epsilon_2^Q \right) \Delta(\epsilon_1^Q) \omega_\mu^{mn} - \epsilon_1^Q \leftrightarrow \epsilon_2^Q \right\} \quad (45)$$

We need on the right-hand side  $(1/2)\bar{\epsilon}_2^Q \gamma^\lambda \epsilon_1^Q R_{\lambda\mu}(Q)$  according to (42), but we have got the expression  $-\{\dots\}$ . Now  $\Delta\omega_\mu^{mn}$  is proportional to  $R(Q)$  but it does not give the exact  $R(Q)$  term we need. The difference is a new constraint

$$\gamma^\mu R_{\mu\nu}(Q) = 0 \quad (\gamma^\mu \equiv \gamma^m e_m^\mu). \quad (46)$$

It can be solved, and then eliminates the conformal gravitino,  $\varphi_\mu^\alpha = \varphi_\mu^\alpha(e, \psi, A, b)$ . Hence there is now also a correction  $\Delta\varphi_\mu$  in the transformation law of the conformal gravitino. The constraint in (46) is an irreducibility constraint; it removes a part from  $R_{\mu\nu}(Q)$  with lower spin. It implies also the second constraint we found before, and can also be written as an antisymmetry condition

$$R_{\mu\nu}(Q) + \frac{1}{2}\epsilon_{\mu\nu}^{\rho\sigma}\gamma_5 R_{\rho\sigma}(Q) = 0, \quad \gamma_{[\mu}R_{\nu\rho]}(Q) = 0. \quad (47)$$

Next we consider the supersymmetry commutator on  $b_\mu$  or  $A_\mu$ . Since  $\delta(\epsilon^Q)b_\mu = (1/2)\bar{\epsilon}\varphi_\mu$  and  $\delta A_\mu = -i\bar{\epsilon}^Q\gamma^5\varphi_\mu$ , we find in these commutators an extra  $\Delta\varphi_\mu^\alpha$  because we already eliminated  $\varphi_\mu$ . We can then compute what  $\Delta\varphi_\mu^\alpha$  should be in order that one gets an Einstein transformation. This yields

$$\Delta(\epsilon^Q)\varphi_\nu = \frac{1}{2}\gamma^\mu\epsilon^Q R_{\mu\nu}{}^D + \frac{i}{4}(\gamma_5\gamma^\mu\epsilon)R_{\mu\nu}(A). \quad (48)$$

On the other hand, we could directly compute  $\delta_{gauge}(\epsilon^Q)\varphi_\mu(e, \psi, A_\mu)$  by the chain rule, and, subtracting  $\delta_{gauge}(\epsilon^Q)\varphi_\mu$ , we would then find what  $\Delta(\epsilon^Q)\varphi_\mu$  really is. The difference between the  $\Delta\varphi_\mu$  which we need and the actual result for  $\Delta\varphi_\mu$  is then the new constraint. The direct evaluation of  $\Delta\varphi_\mu$  is laborious, and a much simpler method is to vary instead the constraint  $\gamma^\mu R_{\mu\nu}(Q) = 0$ , and to find  $\Delta(\epsilon^Q)\varphi_\mu$  by requiring that this variation vanishes. We did this already for  $\Delta\omega_\mu^{mn}$  where we varied  $R(P)$ . Variation of  $\gamma^\mu R_{\mu\nu}(Q) = 0$  leads then to

$$\begin{aligned} \Delta\varphi_\nu = & \frac{1}{6}\gamma^\mu\epsilon R_{\mu\nu}(D) + \frac{1}{24}\gamma_\nu^{\mu\lambda}R_{\mu\lambda}(D) - \frac{1}{4}(\gamma^\mu\gamma_5\epsilon)R_{\mu\nu}(A) - \frac{i}{16}(\gamma_\nu\gamma^{\mu\lambda}\gamma_5\epsilon)R_{\mu\lambda}(A) \\ & - \frac{1}{12}\gamma^\mu\gamma_{mn}\epsilon R_{\mu\nu}{}^{mn}(M) - \frac{1}{48}\gamma_\nu^{\mu\lambda}\gamma_{mn}R_{\mu\lambda}{}^{mn}(M). \end{aligned} \quad (49)$$

Laborious but straightforward algebra then yields for  $\Delta\varphi_\mu$  (needed)  $-\Delta\varphi_\mu$  (as in 49) the following new constraint

$$R_{\mu\nu}{}^{mn}(M)e_n^\nu e_{m\rho} + R_{\mu\rho}(D) + \bar{\psi}^\lambda\gamma_\mu R_{\rho\lambda}(Q) = 0. \quad (50)$$

The term with  $R(Q)$  term is a “supercovariantization”; it is present to remove  $\partial_\mu\epsilon$  terms in the variation of this constraint. (Since  $\Delta\omega \sim \epsilon\gamma R(Q)$  and  $R_{\mu\nu}{}^{mn}(M) \sim \partial_\mu\omega_\nu - \partial_\nu\omega_\mu$ ,

the  $R(Q)$  terms in (50) have the structure  $\bar{\psi}\gamma R(Q)$ .) By taking the part antisymmetric in  $\mu\rho$  one finds a duality constraint

$$R_{\mu\nu}(D) + \frac{i}{4}\epsilon_{\mu\nu}{}^{\rho\sigma}R_{\mu\nu}(A) = 0 \quad (51)$$

(we used  $\gamma_{[\mu}R_{\nu\rho]}(Q) = 0$ ).

Since there are no independent fields left, there are no further constraints. Since all constraints are invariant under  $K, S, M, D, A$  and Einstein symmetry, the transformation laws for these symmetries are without correction terms  $\Delta h_\mu^A$ .

We conclude that the action of conformal  $N = 1$  supergravity is invariant under  $K, S, M, D, A, Q$  and Einstein gauge transformations. The action is given in (40). The physical fields are  $e_\mu^m, \psi_\mu^\alpha, A_\mu$  and  $b_\mu$ , but  $b_\mu$  cancels in the action. The constraints are

$$\left\{ \begin{array}{l} R_{\mu\nu}{}^m(P) = 0, \quad \gamma^\mu R_{\mu\nu}(Q) = 0, \\ R_{\mu\nu}(M) + R_{\mu\nu}(D) + \bar{\psi}^\lambda \gamma_\mu R_{\nu\lambda}(Q) = 0, \\ R_{\mu\nu}(D) + \frac{i}{4}\tilde{R}_{\mu\nu}(A) = 0. \end{array} \right. \quad (52)$$

These constraints are field equations in ordinary supergravity, but here they are only constraints, not field equations. They are imposed by hand from the outside and define the theory. So it makes a difference whether one imposes them or not, and we do impose them. The “Einstein equation” implies the duality constraint  $R(D) \sim \tilde{R}(A)$ , and this enables one to write the Yang-Mills action for  $A_\mu$  in an affine form as  $\epsilon^{\mu\nu\rho\sigma}R_{\mu\nu}(A)R_{\rho\sigma}(D)$ . In one respect simple conformal supergravity is almost too simple: one does not need auxiliary fields.

## 6 Constraints in superspace

The geometry of Einstein space is given by vielbein fields  $e_\mu^m$  and connections  $\omega_\mu^{mn}$ . In supergravity one uses supervielbein fields

$$E_\Lambda^M(\xi, \theta); \quad \Lambda = (\mu, \alpha), \quad M = (m, a) \quad (53)$$

so there are both curved fermionic indices  $\alpha = 1, 4$  and flat fermionic indices  $a = 1, 4$ . The connection is a supervector, but it is only Lorentz-algebra valued

$$\Omega_\Lambda^{mn} = (\Omega_\mu^{mn}, \Omega_\alpha^{mn}) \quad (54)$$

In terms of these two geometrical objects one can define supertorsions  $T_{MN}{}^P$  and super-curvatures  $R_{MN}{}^{mn}$  as follows<sup>7</sup> [5, 6]

$$\{\mathcal{D}_M, \mathcal{D}_N\} = T_{MN}{}^P \mathcal{D}_P + R_{MN}{}^{mn} J_{mn} \quad (55)$$

where  $\mathcal{D}_M = D_M + \Omega_M^{mn} J_{mn}$ , with  $J_{mn}$  the Lorentz generators, and  $D_M = (\partial_m, D_\alpha)$ . The derivative  $D_\alpha$  is the rigid supersymmetry derivative  $D_\alpha = \frac{\partial}{\partial \theta^\alpha} - \theta^\beta \gamma_{\beta\alpha}^\mu \frac{\partial}{\partial x^\mu}$ . We switch at this point from 4-component flat spinor indices  $a = 1, 4$  to 2-component flat spinor indices  $A = 1, 2$  and  $\dot{A} = 1, 2$ . The following is a list of constraints which yield  $N = 1$  supergravity. First there are constraints which express the connections into the supervielbeins, and also the bosonic supervielbein  $E_m{}^\Lambda$  in terms of the fermionic supervielbein  $E_\alpha{}^\Lambda$

$$T_{AB}{}^C = 0, \quad T_{A\dot{B}}{}^{\dot{C}} = 0 \quad (56)$$

$$T_{mn}{}^r = 0, \quad T_{A\dot{B}}{}^m = -i\sigma_{A\dot{B}}^m \quad (57)$$

$$T_{Amn}(\bar{\sigma}^{mn})^{\dot{B}}{}_{\dot{C}} = 0 \quad (58)$$

(the matrices  $\bar{\sigma}^{mn}$  are Lorentz generators). Then there are so-called representation-preserving constraints

$$T_{AB}{}^{\dot{C}} = T_{AB}{}^m = 0 \quad (59)$$

They are needed to be able to define “chiral superfields”, superfields which contain matter (quarks, Higgs bosons, leptons). They are defined by  $\mathcal{D}_A \bar{\phi} = 0$ , and so one has also  $\{\mathcal{D}_B, \mathcal{D}_A\} \bar{\phi} = 0$ , and since  $J^{mn} \bar{\phi} = 0$  on a scalar superfield, one needs for consistency the representation preserving constraints. Finally there are the conformal (sometimes

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<sup>7</sup>In early work, Arnowitt, Nath and Zumino [30] considered a kind of super Lorentz algebra, with generators  $M^{MN}$  instead of  $M^{mn}$ . They wrote down Einstein equations where all vector indices became supervector indices, but this theory contained higher spin fields and ghosts, and was abandoned. In supergravity one only allows the Lorentz group.



called nonconformal) constraints. They turn the superspace theory from a conformal supergravity into a superPoincarè or super anti-de Sitter supergravity. There are various ways of choosing these. One way is

$$T_{Am}{}^n = 0. \quad (60)$$

The transformation laws of tensors in curved space or superspace are usually treated with tensor calculus, according to which a field at  $x$  is related to a field at  $x'$ , for example

$$g'_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} g_{\rho\sigma}(x) \quad (61)$$

However, there exists a formalism which is much more like Yang-Mills theory, due to Siegel and Gates [6]. In Yang-Mills theory a gauge transformation of a scalar field is given by

$$\phi'(x) = e^{U(x)} \phi(x), \quad U(x) = U^a(x) T_a \quad (62)$$

where  $T_a$  are constant matrices acting on the indices of  $\phi(x)$ , and  $U^a(x)$  are the arbitrary local gauge parameters. For supergravity one can write a general supercoordinate transformation as  $Z \rightarrow Z' = Z'(Z)$  where  $Z = (x^\mu, \theta^\alpha)$ . A scalar field transforms then as

$$\phi'(Z) = e^{U(Z)} \phi(Z) \quad (63)$$

$$U(Z) = U^\alpha(Z) D_\alpha + U^m(Z) \partial/\partial x^m \quad (64)$$

The usual definition of the transformation rule of a scalar field is

$$\phi'(Z') = \phi(Z) \quad (65)$$

but if one writes  $\phi(Z)$  as  $\phi(Z) = e^{-U(Z)} \phi'(Z)$  by inverting (63) one finds a more familiar expression

$$\phi'(Z') = e^{-U(Z)} \phi'(Z) \quad (66)$$

In particular

$$Z' = e^{-U(Z)} Z \equiv Z'(Z) \quad (67)$$

This formula gives the relation between the Yang-Mills parameters  $U(Z)$  for superEinstein transformations, and the more conventional parametrization  $Z' = Z'(Z)$ .

In this way one can write superdiffeomorphisms (or ordinary diffeomorphisms) as Yang-Mills transformations; the only difference with internal symmetries is that one has replaced the matrices  $T_a$  of Yang and Mills by operators  $\frac{\partial}{\partial x^\mu}$  and  $D_\alpha$ . Using this formalism, one can solve the constraints on the supertorsions, and find the unconstrained prepotentials which form the starting point for a path integral description of the quantum theory [6].

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